

Automorphism loci for the moduli space of rational maps

Nikita Miasnikov, Brian Stout, and Phillip Williams

ABSTRACT. Let k be an algebraically closed field of characteristic 0 and \mathcal{M}_d the moduli space of rational maps on \mathbb{P}^1 of degree d over k . This paper describes the automorphism loci $A \subset \text{Rat}_d$ and $\mathcal{A} \subset \mathcal{M}_d$ and the singular locus $\mathcal{S} \subset \mathcal{M}_d$. In particular, we determine which groups occur as subgroups of the automorphism group of some $[\phi] \in \mathcal{M}_d$ for a given d and calculate the dimension of the locus. Next, we prove an analogous theorem to the Rauch-Popp-Oort characterization of singular points on the moduli scheme for curves. The results concerning these distinguished loci are used to compute the Picard and class groups of $\mathcal{M}_d, \mathcal{M}_d^s$, and \mathcal{M}_d^{ss} .

1. Introduction and Main Results

Let k be an algebraically closed field and \mathbb{P}^1 the projective line over k . If one fixes homogeneous coordinates X, Y on \mathbb{P}^1 , then any rational map $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree $d > 1$ can be realized as a pair of homogeneous polynomials of degree d in X and Y

$$\phi(X, Y) = [F(X, Y) : G(X, Y)]$$

where F, G have no common roots in k .

Fix a degree $d > 1$. The collection of all such pairs of homogeneous polynomials $[F : G]$ can be naturally parametrized as the projective space \mathbb{P}^{2d+1} where

$$[F : G] = [a_0X^d + a_1X^{d-1}Y + \cdots + a_dY^d : b_0X^d + b_1X^{d-1}Y + \cdots + b_dY^d]$$

and

$$[F : G] \mapsto [a_0 : a_1 : \cdots : b_d]$$

2010 *Mathematics Subject Classification.* Primary: 37P45; Secondary: 14D22.

Key words and phrases. Arithmetic dynamics, moduli space of rational maps, automorphisms, singularities.

Date of last revision: June 26, 2014.

It is clear that not every such pair determines a rational map of degree d on the projective line. There is a homogeneous polynomial of degree $2d$ defined over \mathbb{Z} in the coefficients a_i, b_j , called the *Macaulay resultant*, which vanishes precisely when F and G have a common root in k . We denote the Macaulay resultant by Res and note that $\text{Res} \in \Gamma(\mathbb{P}^{2d+1}, \mathcal{O}(2d))$. It follows that we can construct the parameter space of rational maps on \mathbb{P}^1 as the complement of the vanishing locus of Res .

DEFINITION 1. The space of rational maps of degree d on \mathbb{P}^1 is $\text{Rat}_d = \mathbb{P}^{2d+1} - V(\text{Res})$.

From elementary algebraic geometry, we can see that Rat_d is an affine variety over k of dimension $2d+1$. We will often identify rational maps ϕ of degree d with points of Rat_d .

The space of rational maps carries a natural PGL_2 action of conjugation. For any $f \in \text{PGL}_2$ and any rational map $\phi \in \text{Rat}_d$ we define the *conjugate* of ϕ as $\phi^f = f^{-1} \circ \phi \circ f$. For technical reasons, we restrict ourselves to the analogous action of SL_2 on Rat_d . Since k is algebraically closed, $\text{PSL}_2 \cong \text{PGL}_2$ and we lose no geometric information.

DEFINITION 2. The moduli space of rational maps of degree d on \mathbb{P}^1 is defined to be the geometric quotient $\mathcal{M}_d = \text{Rat}_d / \text{SL}_2$. We use $\pi : \text{Rat}_d \rightarrow \mathcal{M}_d$ to denote the quotient map.

If two rational maps $\phi, \psi \in \text{Rat}_d$ are conjugate we say they are isomorphic. They have the same dynamics as rational maps on \mathbb{P}^1 .

The moduli space \mathcal{M}_d exists as an affine variety over k of dimension $2d-1$. This space is a coarse moduli space for the moduli problem of rational maps: its points parametrize conjugacy classes of rational maps. This space has been studied as a complex orbifold in dynamics over \mathbb{C} (see Appendix G of [8]) and Silverman studied this space as a geometric quotient scheme over \mathbb{Z} (see [15]). Moduli spaces for dynamics on higher dimension projective spaces were constructed and studied by Levy in [6].

Little is known about the geometry of these spaces. Many, however, hope that by studying the geometry of these spaces that properties uniform for all dynamical systems can be determined, much in the same way that the modular curves $X_0(p), X_1(p)$, which are certain moduli of elliptic curves, were used to prove Mazur's Theorem on uniform boundedness of torsion on elliptic curves defined over \mathbb{Q} .

We give a brief synopsis of the known results. For dynamical systems of degree 2, Silverman proved in [15] that $\mathcal{M}_2 \cong \mathbb{A}^2$ as a scheme

over \mathbb{Z} . Silverman gives an explicit isomorphism map in terms of symmetric functions of the multipliers of the fixed points. Levy has shown in [6] that all \mathcal{M}_d are rational varieties. By geometric invariant theory, \mathcal{M}_d is known to be normal, integral, and connected.

As mentioned previously, the moduli space \mathcal{M}_d is not fine: families of dynamical systems parametrized by a base scheme T do not correspond to morphisms $T \rightarrow \mathcal{M}_d$. One cause of this phenomena is the existence of rational maps with automorphisms.

DEFINITION 3. Let $\phi \in \text{Rat}_d$ be a rational map of degree $d > 1$. We say that $f \in \text{PGL}_2$ is an *automorphism of ϕ* if $\phi^f = \phi$.

The collection of automorphisms for fixed ϕ forms a group, denoted by $\text{Aut}(\phi)$; this group is the stabilizer of the point ϕ for the PGL_2 action of conjugation. It is also clear that if $\phi, \psi \in \text{Rat}_d$ are isomorphic, then their automorphism groups are also isomorphic. It is known that these groups are finite and are bounded in terms of d (see [6]). For a point $[\phi] \in \mathcal{M}_d$ we can define $\text{Aut}([\phi])$ as an abstract group using $\text{Aut}(\phi)$ for any map in the conjugacy class.

A natural question to ask is when a rational map has a non-trivial automorphism and to describe the automorphism locus in \mathcal{M}_d . We denote the locus of automorphisms in Rat_d by A and the corresponding locus in \mathcal{M}_d by \mathcal{A} . Specifically,

$$A = \{\phi \in \text{Rat}_d \mid \text{Aut}(\phi) \neq 1\}$$

and $\mathcal{A} = \pi(A)$. The only explicitly known case is for quadratic rational maps. When $d = 2$, via Silverman's isomorphism $\mathcal{M}_2 \cong \mathbb{A}^2$, the automorphism locus \mathcal{A} is known to be a cuspidal cubic. Furthermore, the only possible automorphism groups are \mathcal{S}_3 , which occurs precisely at the cusp, and $\mathbb{Z}/2\mathbb{Z}$ which occurs at all other points of this curve (see Proposition 4.15 of [14]). For a general $d > 1$, it is known that the only possible automorphisms groups are $\mathbb{Z}/m\mathbb{Z}$, the cyclic group of order m , D_m , the dihedral group of order $2m$, A_4, A_5 , the alternating group on four or five letters, respectively, or S_4 the full symmetric group on four letters. See Example 2.54 of [14].

A second special locus to consider is the singular locus. We denote the singular locus of \mathcal{M}_d by \mathcal{S} , and \mathcal{S} consists of all points in \mathcal{M}_d which have tangent space of dimension strictly larger than $2d - 1$. Again, the only explicitly known case is for quadratic rational maps, where $\mathcal{S} = \emptyset$ since \mathcal{M}_2 is smooth. It would be interesting to determine if these "singular" rational maps have additional dynamical structure.

A third geometric question about the moduli space of rational maps, which is related to the previous two, consists of calculating the Picard

group $\text{Pic}(\mathcal{M}_d)$. When the action of an algebraic group on an affine variety is free, the Picard group of the quotient variety is particularly easy to calculate. Because of the existence of non-trivial automorphisms, the action of SL_2 on Rat_d is not free. However, after determining the automorphism loci $A \subset \text{Rat}_d$ and $\mathcal{A} \subset \mathcal{M}_d$ to be of codimension greater than 1, we compute $\text{Pic}(\mathcal{M}_d)$, $\text{Pic}(\mathcal{M}_d^s)$, and $\text{Pic}(\mathcal{M}_d^{ss})$ by looking at the locus where the action is free.

We now discuss the results of this paper.

We begin by studying the loci of points in \mathcal{M}_d whose automorphism group contains a copy of a cyclic or dihedral group.

DEFINITION 4. Let G be a finite subgroup of PGL_2 .

$$\mathcal{A}(G) = \{[\phi] | \text{Aut}([\phi]) \supseteq G\}$$

In Section 2 we calculate the dimensions of $\mathcal{A}(G)$ when G is cyclic or dihedral, in particular we show that $\mathcal{A}(\mathbb{Z}/m\mathbb{Z}) \neq \emptyset$ if and only if m divides one of d , $d \pm 1$.

In Section 3 we use the decomposition of L. West for the $2d + 2$ -dimensional representation of SL_2 associated with conjugation of rational maps to obtain the following result which characterizes the remaining possible automorphism groups.

PROPOSITION. *Let $d > 1$.*

- (1) $\mathcal{A}(A_4) \neq \emptyset$ if and only if d is odd.
- (2) $\mathcal{A}(S_4) \neq \emptyset$ if and only if d is coprime to 6.
- (3) $\mathcal{A}(A_5) \neq \emptyset$ if and only if d is congruent to one of 1, 11, 19, 21 modulo 30.

Let G be any of the groups A_4 , A_5 , S_4 and suppose d is such that $\mathcal{A}(G) \neq \emptyset$. Then $\mathcal{A}(G)$ is irreducible and

$$\dim \mathcal{A}(G) = \left\lfloor \frac{2d}{|G|} \right\rfloor.$$

where $\lfloor n \rfloor$ denotes the the greatest integer less than or equal to n .

In Section 4 we apply the Luna Slice theorem and our computations of dimensions in Section 2 to prove the following theorem inspired by Rauch-Popp-Oort's characterization of singular points on the moduli space for curves.

THEOREM. *Let $d > 2$. Then the singular locus \mathcal{S} and the automorphism locus \mathcal{A} of \mathcal{M}_d coincide.*

The Luna Slice theorem is an algebro-geometric tool and for us \mathcal{M}_d is a quotient *variety*. However, $\mathcal{M}_d(\mathbb{C})$ equipped with the analytic

topology becomes a topological quotient of $\text{Rat}_d(\mathbb{C})$, when the latter is equipped with the usual topology (cf [12]).

Our strategy is as follows: We call a point ϕ in the automorphism locus *simple* if $\text{Aut}(\phi) \cong \mathbb{Z}/p\mathbb{Z}$ with p prime. We then show the following:

- (1) Simple points in \mathcal{M}_d are algebraically singular and in fact topologically singular, i.e. they do not possess a Euclidean neighborhood in the analytic topology.
- (2) Simple points are dense in the automorphism locus of \mathcal{M}_d .
- (3) Finally we use the fact that algebraic singularities form a closed set.

In Section 5 we use the results of the previous sections to calculate Picard and class groups of \mathcal{M}_d , its projective closure \mathcal{M}_d^{ss} and the intermediate variety \mathcal{M}_d^s . Our main tool for the following theorem is Narahimhan-Drezet's Descent Lemma for vector bundles equipped with group action (see [11]).

THEOREM. *Let $d > 1$ and $\mathcal{M}_d, \mathcal{M}_d^s, \mathcal{M}_d^{ss}$ be the moduli space of rational maps of degree d , stable points in \mathbb{P}^{2d+1} , and semi-stable points in \mathbb{P}^{2d+1} . Then,*

- (1) $\text{Cl}(\mathcal{M}_d) = \mathbb{Z}/2d\mathbb{Z}$ or $\mathbb{Z}/d\mathbb{Z}$ when d is odd and even, respectively.
- (2) $\text{Cl}(\mathcal{M}_d^s) = \text{Cl}(\mathcal{M}_d^{ss}) = \mathbb{Z}$.
- (3) $\text{Pic}(\mathcal{M}_d)$ is trivial.
- (4) $\text{Pic}(\mathcal{M}_d^s) = \text{Pic}(\mathcal{M}_d^{ss}) = \mathbb{Z}$.

Acknowledgment. The authors would like to thank John Milnor for his helpful correspondence and Lloyd West for helpful conversations.

2. Dimension of the automorphism locus

In this section we work over \mathbb{C} , although the conclusions hold for any algebraically closed field of characteristic 0.

We recall the definition of an automorphism of a rational map from Section 1.

DEFINITION 5. Let $\phi \in \text{Rat}_d$ be a rational map of degree $d > 1$. We say that $f \in \text{PGL}_2$ is an *automorphism* if $\phi^f = \phi$.

For every rational function ϕ there is a canonical metric g on $\mathbb{P}^1(\mathbb{C})$ which makes $\mathbb{P}^1(\mathbb{C})$ isometric to the usual 2-dimensional sphere. Any automorphism $f \in \text{Aut}(\phi)$ must fix the metric g , and therefore $\text{Aut}(\phi)$ embeds into $\text{SO}_3(\mathbb{R})$. It is well known that $\text{Aut}(\phi)$ is always finite (Proposition 4.65 of [13]) and the complete list of finite subgroups

of $\mathrm{SO}_3(\mathbb{R})$ are: cyclic order m , dihedral D_m (of order $2m$), and the groups of rotations of platonic solids A_4 (tetrahedron), S_4 (cube and octahedron), and A_5 (dodeca- and icosohedron) (Remark 4.66 [13]).

Thus the groups arising as $\mathrm{Aut}(\phi)$ are all from this list. In this section we examine $\mathcal{A}(G)$ and calculate its dimension for all G . We will use the shorter notation $\mathcal{A}_m = \mathcal{A}(\mathbb{Z}/m\mathbb{Z})$ for the case of a cyclic group of order m .

We start with a well-known lemma, and then proceed case-by-case depending on the subgroup $G \subset \mathrm{PGL}_2$.

LEMMA 1. *Two finite subgroups of PGL_2 which are isomorphic are conjugate. Moreover,*

$$\begin{aligned} \langle \zeta_m z \rangle &\cong \mathbb{Z}/m\mathbb{Z}, & \langle \zeta_m z, 1/z \rangle &\cong D_m, \\ \left\langle -z, i \frac{z+1}{z-1} \right\rangle &\cong A_4, & \left\langle iz, i \frac{z+1}{z-1} \right\rangle &\cong S_4. \end{aligned}$$

PROOF. The first statement is well known and dates back to Klein, [5], and the rest follows immediately. \square

2.1. Cyclic groups. First we look at the case of a cyclic group. The presentation of the material in this section was suggested by John Milnor. Let ζ be a primitive m -th root of unity with $m > 1$. By Lemma 1 every finite cyclic subgroup of $\mathrm{PGL}_2(\mathbb{C})$ of order m is conjugate to $\langle z \mapsto \zeta z \rangle$. If ϕ commutes with $\langle z \mapsto \zeta z \rangle$ then it must map the set $\{0, \infty\} = \mathrm{Fix}(\langle z \mapsto \zeta z \rangle)$ to itself. Hence, ϕ induces a set map $\{0, \infty\} \rightarrow \{0, \infty\}$, leading us to the following definition suggested again by John Milnor.

DEFINITION 6. If $\sigma \in \mathrm{PGL}_2(\mathbb{C})$ is a non-trivial automorphism of a rational function ϕ , we say that σ is an automorphism of ϕ of type t if

$$t + 1 = |\mathrm{Fix}(\phi) \cap \mathrm{Fix}(\sigma)|.$$

Since every automorphism of \mathbb{P}^1 of finite order is conjugate to ζz , it has two fixed points and hence the type t can only be $-1, 0$, or 1 . Let $A_\zeta^t \subset \mathrm{Rat}_d$ (where t is abbreviated to one of $-, 0, +$ rather than $-1, 0, 1$) be the locus of all maps for which $\langle z \mapsto \zeta z \rangle$ is an automorphism of type t . We will now characterize A_ζ^t .

LEMMA 2. *A rational map ϕ commutes with the linear map $z \mapsto \zeta z$ if and only if it has the form $\phi(z) = z\psi(z^m)$ for some rational function ψ .*

PROOF. Setting $\eta(z) = \phi(z)/z$, note that

$$\eta(\zeta z) = \phi(\zeta z)/(\zeta z) = \phi(z)/z = \eta(z).$$

It follows easily that $\eta(z) = \psi(z^m)$ for a uniquely defined rational function ψ . The converse is a straightforward calculation. \square

Let $d \geq 2$ be the degree of ϕ and let $d' \geq 1$ be the degree of ψ . By Lemma 2, set

$$\psi(u) = \frac{\alpha u^{d'} + \cdots + \beta}{\gamma u^{d'} + \cdots + \delta}$$

so that

$$\psi(\infty) = \alpha/\gamma \text{ and } \psi(0) = \beta/\delta.$$

It follows easily that

$$\phi(0) = \infty \iff \psi(0) = \infty \iff \delta = 0,$$

with $\phi(0) = 0$ whenever $\delta \neq 0$. Similarly

$$\phi(\infty) = 0 \iff \psi(\infty) = 0 \iff \alpha = 0,$$

with $\phi(\infty) = \infty$ whenever $\alpha \neq 0$.

The discussion can now be divided into three cases according to the type of the map $z \mapsto \zeta z$ as an automorphism of ϕ .

Type 1. If $\alpha \neq 0$, and $\delta \neq 0$, so that 0 and ∞ are fixed points of ϕ , then it follows easily that

$$d = md' + 1,$$

$$\dim(A_\zeta^+) = 2d' + 1.$$

Type -1. If $\alpha = \delta = 0$, so that $\{0, \infty\}$ is a period two orbit for ϕ , then canceling a factor of z from the numerator and denominator of ϕ we see that $d < md'$, and it follows easily that

$$d = md' - 1,$$

$$\dim(A_\zeta^-) = 2d' - 1.$$

Type 0. In the intermediate case where just one of the two coefficients α and δ is zero, so that $\phi(0) = \phi(\infty) \in \{0, \infty\}$, the locus A_ζ^0 has two irreducible components which are conjugate via $z \mapsto 1/z$:

$$\{\phi|\phi(\zeta z) = \zeta\phi(z), \phi(0) = \phi(\infty) = 0\} \subset \text{Rat}_d$$

and

$$\{\phi|\phi(\zeta z) = \zeta\phi(z), \phi(0) = \phi(\infty) = \infty\} \subset \text{Rat}_d$$

and a similar argument shows that for each component,

$$d = md',$$

$$\dim(A_\zeta^0) = 2d'$$

Let $\mathcal{A}_m^t = \pi(A_m^t)$ be the locus in \mathcal{M}_d of points parametrizing conjugacy classes which admit an automorphism of order m and type t . Then

$$\dim(\mathcal{A}_m^t) = \dim(A_\zeta^t) - 1,$$

since the group $\{z \mapsto cz \mid c \in k\}$ acts on A_ζ^t and all fibers of the surjective map $A_\zeta^t \rightarrow \mathcal{A}_m^t$ are 1-dimensional, being finite unions of orbits under this action.

This discussion leads to the following proposition.

PROPOSITION 3. *The space Rat_d admits a map with an automorphism of order m of type t if and only if m is a divisor of $d - t$. In such a case the resulting locus in \mathcal{M}_d of all maps having an automorphism of order m and type t is closed and irreducible of dimension*

$$\dim(\mathcal{A}_m^t) = \frac{2(d - t)}{m} + t - 1$$

PROOF. Everything has been established above except that the locus is closed. This follows from Luna Slice theorem (cf [2]). \square

REMARK 1. Except for the case $m = 2$ and d odd, an automorphism of order m can occur only in one type. In this case $\mathcal{A}_m = \mathcal{A}_m^t$ depending on the type t that does occur. When d is odd, automorphisms of order 2 can occur in types $+1, -1$. Thus for odd d , $\mathcal{A}_2 = \mathcal{A}_2^- \cup \mathcal{A}_2^+$.

REMARK 2. It is easy to see that if $2 < m < n$,

$$d - 1 = \dim(\mathcal{A}_2) > \dim(\mathcal{A}_m) \geq \dim(\mathcal{A}_n).$$

COROLLARY 4. *For $d \geq 1$, the codimension of the automorphism loci $A \subset \text{Rat}_d$ and $\mathcal{A} \subset \mathcal{M}_d$ is $d - 1$.*

PROOF. This follows immediately for $\mathcal{A} \subset \mathcal{M}_d$. For A , we have that because A is SL_2 invariant, closed, and contained in the stable locus, \mathcal{A} is a geometric quotient of A by the action of SL_2 . It follows that $\text{codim}(A, \text{Rat}_d) = \text{codim}(\mathcal{A}, \mathcal{M}_d)$. \square

2.2. Dihedral groups. We now examine the locus, $\mathcal{A}(D_m)$, of those $[\phi]$ for which $\text{Aut}(\phi)$ contains a copy of the dihedral group D_m of order $2m$. Recall, that by Lemma 1, every dihedral group D_m inside $\text{PGL}_2(\mathbb{C})$ is conjugate to $\langle \zeta_m z, 1/z \rangle$.

LEMMA 5. *A rational function ϕ commutes with both $\zeta_m z$ and $1/z$ if and only if it can be written in the form $\phi = z\psi(z^m)$ where ψ commutes with $1/z$.*

PROOF. By Lemma 2 ϕ can be written in the form $\phi(z) = z\psi(z^m)$ if and only if it commutes with $\zeta_m z$. Now, $z\psi(z^m)$ commutes with $1/z$ if and only if $z\psi(z^m) = \frac{1}{1/z\psi(1/z^m)} \iff \phi(z^m) = \frac{1}{\psi(1/z^m)} \iff \psi(u) = \frac{1}{\psi(1/u)}$ if and only if $1/u$ commutes with ψ . \square

PROPOSITION 6. *The dimensions of $\mathcal{A}(D_m)$ are as follows:*

$$\begin{aligned} (t = 1) \text{ if } m|d-1, \dim \mathcal{A}(D_m) &= \frac{d-1}{m} \\ (t = 0) \text{ if } m|d, \mathcal{A}(D_m) &= \emptyset \\ (t = -1) \text{ if } m|d+1, \dim \mathcal{A}(D_m) &= \frac{d+1}{m} - 1 \end{aligned}$$

PROOF. Suppose ϕ commutes with both ζ_m and $1/z$. Then we have from Lemma 5 that $\phi = z\psi(z^m)$ where ψ commutes with $1/z$. Let $[a_0, \dots, a_{d'}, b_0, \dots, b_{d'}]$ be the coefficients of ψ .

In the $t = 1$ case, the degree of ψ is $d' = \frac{d-1}{m}$, and this gives $2d' + 2 = 2\frac{d-1}{m} + 2$ coefficients. Because ψ commutes with $1/z$, the coefficients of ψ satisfy the additional $d' + 1$ homogeneous equations $a_0 = \lambda b_{d'}, a_1 = \lambda b_{d'-1}, \dots, a_{d'} = \lambda b_0$ where $\lambda = \pm 1$. Each of these independent equations cuts the dimension down by one. Therefore the dimension of the locus in Rat_d in this case is:

$$2d' + 2 - (d' + 1) - 1 = d' = \frac{d-1}{m}$$

In the $t = 0$ case, exactly one of $a_0, b_{d'}$ vanishes, which is impossible for a map that commutes with $1/z$. Therefore, $\mathcal{A}(D_m) = \emptyset$.

In the $t = -1$ case, the degree of ψ is $d' = \frac{d+1}{m}$, and this gives $2d' + 2 = 2\frac{d+1}{m} + 2$ coefficients. That $1/z$ is an automorphism of ψ gives the same relations as in the $t = 1$ case; also we have that $a_0 = 0, b_{d'} = 0$, so the first relation $a_0 = \pm b_{d'}$ is redundant. This gives a total of $d' + 2$ independent equations. Therefore the dimension of the locus in Rat_d in this case is

$$2d' + 2 - (d' + 2) - 1 = \frac{d+1}{m} - 1$$

Since there is no infinite family in PGL_2 which commutes with both $\zeta_m z$ and $1/z$, the map from the locus $\mathcal{A}(D_m)$ in Rat_d to $\mathcal{A}(D_m)$ is finite to one, and the result follows. \square

REMARK 3. Note that for $m = 2$ the cases (± 1) agree with each other.

2.3. Automorphism locus in \mathbb{P}^{2d+1} .

PROPOSITION 7. *For $d \geq 1$ the codimension of the automorphism locus in \mathbb{P}^{2d+1} is $d - 1$.*

PROOF. One can prove this directly in a way similar to the proof for Rat_d . However here is another proof. We have already proved the corresponding statement for Rat_d (Corollary 4), so for any irreducible component Z which meets Rat_d , $\text{codim} Z \geq d - 1$. Let Z be an irreducible component of $\text{Aut}(\mathbb{P}^{2d+1})$ entirely inside $V(\text{Res})$. Let

$$m = \min\{\deg \gcd(F, G) \mid (F : G) \in Z\}$$

and let $W = H^0(\mathbb{P}^1, \mathcal{O}(m))$, the space of m -forms in two variables. Consider the morphism:

$$f : \mathbb{P}(W) \times \mathbb{P}^{2(d-m)+1} \rightarrow V(\text{Res})$$

given by

$$([H], (F : G)) \mapsto (HF : HG)$$

It is not hard to see that $\mathbb{P}(W) \times \text{Rat}_{d-m}$ carries a PGL_2 -action and that the map f is equivariant for this action. Since the image of f restricted to $\mathbb{P}(W) \times \text{Rat}_{d-m}$ consists of all pairs $(F : G)$ with $\deg(\gcd(F, G)) = m$ (i.e. common zeros of exact order m), it follows that $f^{-1}(f(\mathbb{P}(W) \times \text{Rat}_{d-m})) = \mathbb{P}(W) \times \text{Rat}_{d-m}$.

We claim the restriction of f to $\mathbb{P}(W) \times \text{Rat}_{d-m}$ is injective. Suppose $([H_1], (F_1 : G_1)), ([H_2], (F_2 : G_2))$ both map to $(H_1 F_1 : H_1 G_1) = (H_2 F_2 : H_2 G_2)$. The, for some $\lambda \in k^\times$, $H_1 F_1 = \lambda H_2 F_2$ and $H_1 G_1 = \lambda H_2 G_2$. These equalities imply that $H_1 | H_2 F_2$ and $H_1 | H_2 G_2$, but $(F_2 : G_2) \in \text{Rat}_{d-m}$ means they have no common zeros. It follows that $H_1 | H_2$, and since their degrees are the same, $[H_1] = [H_2]$ in $\mathbb{P}(W)$. After canceling in the previous equations we have that $(F_1 : G_1) = (F_2 : G_2)$ and the restriction is injective.

Now, the image of f contains Z and Z meets $f(\mathbb{P}(W) \times \text{Rat}_{d-m})$. From injectivity it follows that

$$f(\text{Aut}(\mathbb{P}(W) \times \text{Rat}_{d-m})) = \text{Aut}(f(\mathbb{P}(W) \times \text{Rat}_{d-m}))$$

and we also have

$$\text{Aut}(\mathbb{P}(W) \times \text{Rat}_{d-m}) \subseteq \mathbb{P}(W) \times \text{Aut}(\text{Rat}_{d-m})$$

Thus we get

$$\begin{aligned} \dim Z &= \dim Z \cap f(\mathbb{P}(W) \times \text{Rat}_{d-m}) \\ &\leq \dim \text{Aut} f(\mathbb{P}(W) \times \text{Rat}_{d-m}) \\ &\leq \dim \text{Aut}(\mathbb{P}(W) \times \text{Rat}_{d-m}) \\ &\leq \dim \mathbb{P}(W) \times (\text{Aut} \text{Rat}_{d-m}) \\ &= \begin{cases} m + (d - m) + 2 = d + 2 & \text{for } m < d \\ d + 1 & \text{for } m = d \end{cases} \end{aligned}$$

We remark that the first equality arises because f is projective, and hence closed, and so $\overline{f(\mathbb{P}(W) \times \text{Rat}_{d-m})} = f(\mathbb{P}(W) \times \mathbb{P}^{2(d-m)+1})$. \square

Given an element $g \in \text{SL}_2$, a point $\phi \in \mathbb{P}^{2d+1}$ fixed by g comes from an eigenvector of multiplication by g in \mathbb{A}^{2d+2} . Specifically, g acts on the stalk of $\mathcal{O}(1)$ and $\mathcal{O}(-1)$ at ϕ via multiplication by the eigenvalue.

LEMMA 8. *Let $\eta \neq 0$. Suppose $\Phi = (F, G) \in \mathbb{A}^{2d+2}$ is an eigenvector for the linear transformation given by the action of $\begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix}$ and λ the associated eigenvalue. Write*

$$(F, G) = (a_0, \dots, a_d, b_0, \dots, b_d).$$

Then,

- (1) *If k is such that $a_k \neq 0$, then $\lambda = \eta^{d-2k-1}$.*
- (2) *If k is such that $b_k \neq 0$, then $\lambda = \eta^{d-2k+1}$.*

PROOF. Under the hypothesis of the lemma it follows that:

$$(\eta^{-1}F(\eta X, \eta^{-1}Y), \eta G(\eta X, \eta^{-1}Y)) = \lambda(F, G)$$

and therefore that

$$(a_0\eta^{d-1}, \dots, \frac{a_d}{\eta^{d+1}}, b_0\eta^{d+1}, \dots, \frac{b_d}{\eta^{d-1}}) = (\lambda a_0, \dots, \lambda a_d, \lambda b_0, \dots, \lambda b_d)$$

If $a_k \neq 0$, then one can equate the corresponding coordinates and solve for λ , resulting in the first equation. In a similar fashion one obtains the equation when $b_k \neq 0$. \square

COROLLARY 9. *Suppose $\phi \in \text{Rat}_d$ and $\sigma \in \text{PGL}_2$ an automorphism of ϕ of order $m > 1$ and type t . Let $d' = (d - t)/m$. Let $g \in \text{SL}_2$ represent σ and assume that order of g is $2m$. Then g acts on the stalk of $\mathcal{O}(1)$ at ϕ via multiplication by a root of unity whose order s is as follows:*

Case $t = \pm 1$: $s = 1$ if d' is even, and $s = 2$ if d' is odd.

Case $t = 0$: $s = m$ if d is odd and $s = 2m$ if d is even.

PROOF. After an appropriate conjugation we can assume that

$$\sigma(z) = \zeta z,$$

where ζ is m -th root of unity. Then we have the following: If $t = 1$, $a_0 \neq 0$. If $t = -1$, $b_0 \neq 0$. If $t = 0$, either $a_0 \neq 0$ or $b_0 \neq 0$, with the same result. \square

The above lemma and corollary will help us to compute the Picard and class groups of \mathcal{M}_d , \mathcal{M}_d^s , and \mathcal{M}_d^{ss} in Section 5.

3. Automorphism loci for groups of rotations of platonic solids

In this section we use L. West's decomposition to calculate dimension formulas for automorphism loci for tetrahedral group A_4 , octahedral group S_4 and the icosahedral group A_5 .

Let V be the vector space k^2 . Then SL_2 acts naturally on the left on V by multiplication. Let X, Y be the coordinate functions on k^2 corresponding to the standard basis, and let $R = k[X, Y]$. Let R_n denote the n -th homogeneous component of R . Since R is the coordinate ring of V , the action of SL_2 on V induces a right SL_2 -action on R by pre-composition, where if

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $F \in R$, then $F^g = F(aX + bY, cX + dY)$. This action preserves grading hence there is a (right) action on each homogeneous component R_n . Let W be the space of morphisms $V \rightarrow V$ given by a pair of homogeneous polynomials of degree d in X and Y . Then SL_2 acts on the right on W by conjugation: $(\phi^A)(v) = A^{-1}\phi(Av)$.

THEOREM 10 (L. West [16]). *The following is an isomorphism of SL_2 representations:*

$$g : W \rightarrow R_{d-1} \oplus R_{d+1}$$

$$g(F_1, F_2) = \left(\frac{\partial F_1}{\partial X} + \frac{\partial F_2}{\partial Y}, YF_1 - XF_2 \right)$$

whose inverse is:

$$g^{-1}(H, J) = \frac{1}{d+1} \left(XH + \frac{\partial J}{\partial Y}, YH - \frac{\partial J}{\partial X} \right)$$

PROOF. Morphisms $V \rightarrow V$ can be canonically (independently of choice of coordinates on V) identified with vector fields on V , by which we mean elements of $T := \mathrm{Hom}_R(\Omega_{R/k}^1, R)$. Namely for each vector field $F = F_1\partial/\partial X + F_2\partial/\partial Y \in T$, we get the morphism $V \rightarrow V$ given by $(X, Y) \rightarrow (F_1, F_2)$. Let us now endow V with the volume form \langle, \rangle given by $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1y_2 - y_1x_2$. This form is preserved by the SL_2 action. To prove that g is an SL_2 -morphism, we will define g in terms of quantities preserved by the SL_2 -action.

$$g(F) = (\nabla \cdot F, \langle F, r \rangle)$$

$$g^{-1}(H, J) = \frac{1}{d+1} (Hr + (dJ)^\#)$$

where $r = (X, Y)$, $v^b = \langle v, - \rangle$, \sharp is the inverse of b , and where we identify vector fields on k^2 with maps $k^2 \rightarrow k^2$.

□

NOTATION 1. By \bar{g} we denote the PGL_2 -isomorphism

$$\bar{g} : \mathbb{P}^{2d+1} \rightarrow \mathbb{P}(R_{d-1} \oplus R_{d+1})$$

which is induced by g .

The following is a key lemma about the image of Rat_d under the above decomposition \bar{g} .

LEMMA 11. *Let \mathbb{G}_m act on $R_{d-1} \oplus R_{d+1}$ via*

$$t \cdot (H, J) = (tH, t^{-1}J).$$

This action induces a \mathbb{G}_m -action on $\mathbb{P}(R_{d-1} \oplus R_{d+1})$. This action commutes with the PGL_2 action. Hence PGL_2 -stabilizers are identical for all points within a \mathbb{G}_m -orbit. The \mathbb{G}_m orbit of $[(H, J)]$ meets $\bar{g}(\mathrm{Rat}_d)$ if and only if no multiple zero of J is also a zero of H .

PROOF. The commutativity with the PGL_2 action is immediate. Suppose that a multiple zero of J is also a zero of H . Without loss of generality, we may suppose this zero to be the coordinate Y , and we have

$$g^{-1}(tH, t^{-1}J) = \frac{1}{d+1} \left(tXH + t^{-1} \frac{\partial J}{\partial Y}, tYH - t^{-1} \frac{\partial J}{\partial X} \right)$$

for all t . From this formula it is clear that Y is a common root for any t and thus $\mathbb{G}_m \cdot [(H, J)]$ does not meet $\bar{g}(\mathrm{Rat}_d)$.

Conversely, suppose

$$[(F, G)] = \frac{1}{d+1} \left[\left(tXH + t^{-1} \frac{\partial J}{\partial Y}, tYH - t^{-1} \frac{\partial J}{\partial X} \right) \right] \notin \mathrm{Rat}_d$$

for all t . This means that F and G have a common zero for all t . Now, a well known identity says that

$$X \frac{\partial J}{\partial X} + Y \frac{\partial J}{\partial Y} = (d+1)J$$

Thus if (a, b) is a common zero of F and G , then evaluating the above expression at (a, b) gives that $t^2aH = -\frac{\partial J}{\partial Y}|_{(a,b)}$ and $t^2bH = \frac{\partial J}{\partial X}|_{(a,b)}$. Plugging this into the above identity gives that (a, b) is also a zero of J . Assuming J is not the zero polynomial (in which case we have nothing to show), this implies that at least one of the common zeros of F and G occurs infinitely often as t varies. Because the PGL_2 action

commutes with the \mathbb{G}_m action, we may take this zero to be $[0 : 1]$ —i.e. Y is a factor of F and G for infinitely many t . Now, if H and $\frac{\partial J}{\partial Y}$ are not both divisible by Y , it would be impossible for $tXH + t^{-1}\frac{\partial J}{\partial Y}$ to be divisible by Y for infinitely many t , and so we have that Y divides both H and $\frac{\partial J}{\partial Y}$. Similarly Y must divide $\frac{\partial J}{\partial X}$. It follows that Y is a multiple root of J . \square

Lemma 11 implies the following interesting corollary, which we will not need in the remainder of the paper.

COROLLARY 12. *Let G be non-cyclic and suppose that $\dim \mathcal{A}(G) = 0$. Then if $\text{Aut}(\phi)$ contains a copy of G , then ϕ must have $\deg(\phi) + 1$ distinct fixed points with all multipliers equal to $\deg(\phi)$.*

LEMMA 13. *Let m, k be positive integers, η a primitive $2m^{\text{th}}$ root of 1, and F a binary form of degree k for which 0 and ∞ are not multiple zeros and suppose F is an eigenvector for the action of $\begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix}$ with eigenvalue λ . Then λ is as in Table 1.*

Fixed zeros	$F(0) \neq 0$	$F(0) = 0$
$F(\infty) \neq 0$	$m k$ $\lambda = (-1)^{\frac{k}{m}}$	$m k - 1$ $\lambda = (-1)^{\frac{k-1}{m}} \eta$
$F(\infty) = 0$	$m k - 1$ $\lambda = (-1)^{\frac{k-1}{m}} \eta^{-1}$	$m k - 2$ $\lambda = (-1)^{\frac{k-2}{m}}$

TABLE 1. Eigenforms without 0 and ∞ as multiple zeros

PROOF. Write $F(X, Y) = a_0X^k + a_1X^{k-1}Y + \dots + a_kY^k$. Because F is an eigenvector we have that

$$\begin{aligned} F(\eta X, \eta^{-1}Y) &= a_0\eta^k X^k + a_1\eta^{k-2}X^{k-1}Y + \dots + \frac{a_k}{\eta^k}Y^k \\ &= \lambda a_0X^k + \lambda a_1X^{k-1}Y + \dots + \lambda a_kY^k \end{aligned}$$

We write $F(0) := F(0, 1)$ and $F(\infty) := F(1, 0)$. If $F(0) = 0$, then $a_k = 0$, but because 0 is not a multiple zero, $a_{k-1} \neq 0$. Similarly, if $F(\infty) = 0$, then $a_0 = 0$, but since ∞ is not a multiple zero, $a_1 \neq 0$.

The four cases in the table now correspond to four systems of linear equations. If $F(0), F(\infty) \neq 0$, then $a_0, a_k \neq 0$. Equating first and last coefficients and canceling a_0 and a_k gives the system $\lambda = \eta^k$ and

$\lambda = 1/\eta^k$. Since η is a $2m^{\text{th}}$ root of unity, it follows that $m|k$ and $\lambda = (-1)^{\frac{k}{m}}$.

Similarly, if $F(0) = F(\infty) = 0$, then $a_0 = a_k = 0$ and we can equate the second and second to last coefficients giving the system $\lambda = \eta^{k-2}$ and $\lambda = 1/\eta^{k-2}$. In this case $m|k-2$ and $\lambda = (-1)^{\frac{k-2}{m}}$.

The other two cases are symmetrical. Suppose $F(0) = 0$ and $F(\infty) \neq 0$. Then $a_k = 0, a_{k-1} \neq 0$, and $a_0 \neq 0$. Equating first and second to last coefficients and canceling a_0 and a_{k-1} gives $\lambda = \eta^k$ and $\lambda = 1/\eta^{k-2}$. Solving for η gives $1 = \eta^{2k-2}$, hence $m|(k-1)$ and $\lambda = (-1)^{\frac{k-1}{m}}\eta$. If $F(0) \neq 0, F(\infty) = 0$ the calculation is identical, but $\lambda = (-1)^{\frac{k-1}{m}}\eta^{-1}$. \square

As described in the beginning of this section, there is a natural SL_2 -action on R_n which induces a $\text{PSL}_2 = \text{PGL}_2$ action on $\mathbb{P}(R_n)$, which we can consider as a PGL_2 action on the group of divisors: if $F \in R_n$ and $D = \text{div}(F)$, then $D^g = \text{div}(F^g)$. Thus if $D = \sum n_i(P_i)$, $D^g = \sum n_i(g^{-1} \cdot P_i)$.

For a divisor D on \mathbb{P}^1 and a finite group $G \subset \text{PGL}_2$, then we can consider the orbit of D under G . If $p \in \mathbb{P}^1$, then we can consider p as a divisor and speak of its orbit under this action. This orbit consists of at most $|G|$ points. If the orbit has less than $|G|$ points, then D has non-trivial stabilizer under this action.

LEMMA 14. *Let $G \subset \text{PGL}_2$ be a finite subgroup of order k , k even. Let $P \in \mathbb{P}^1$. Let D be the divisor*

$$D = \sum_{\sigma \in G} P^\sigma$$

Let $\sigma_0 \in G$ and let g be a pre-image of σ_0 in SL_2 and denote by m the order of σ_0 . Let $F \in R_k$ be such that $\text{div}(F) = D$. Then the action of g on F is multiplication by $(-1)^{\frac{k}{m}}$, and in particular, if G is one of A_4, S_4, A_5 , the pre-image of G in SL_2 fixes F .

PROOF. Let n be the order of the stabilizer of P in G . Then for any $\sigma \in G$, the stabilizer of P^σ in G is also order n , so $D = nD'$ for some divisor D' with no coefficients greater than 1. D' is, in fact, the support of D . The degree of D' is $\frac{k}{n}$.

Now let F be a homogeneous form such that $\text{div}(F) = D$ and let σ_0 and g be as in the statement of the lemma. We can write $F = H^n$ where H is a homogenous form with no multiple zeros and $\text{div}(H) = D'$. Since D is clearly fixed by G , the action of g on F must be multiplication by a scalar λ . The same is true for D' and H : g acts on H by a scalar λ' and we have that $(\lambda')^n = \lambda$.

Now, with Lemma 13 in mind, let η be a primitive $2m^{th}$ root of unity and consider the matrix:

$$\begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix} \in \mathrm{SL}_2$$

Conjugating g to this matrix does not change the eigenvalues λ and λ' , and so without loss of generality we may assume g has this form. Note also that the two lifts of σ_0 to $g \in \mathrm{SL}_2$ give the same eigenvalue λ for F because k is even. Now apply Lemma 13 to H . For the top left case of this lemma, the eigenvalue of H is $\lambda' = (-1)^{\frac{k/n}{m}}$ so that $\lambda = (-1)^{\frac{k}{m}}$. For the bottom right case, first note that in this case 0 is a zero of H , and also a fixed point of σ_0 (after our conjugation). This means that σ_0 is in the stabilizer of 0, which, being a root of H , is order n . Thus $m|n$. Then $(-1)^{\frac{k/n-2}{m}}$ raised to the n^{th} power gives $\lambda = (-1)^{\frac{k-2n}{m}} = (-1)^{\frac{k}{m}}(-1)^{\frac{-2n}{m}} = (-1)^{\frac{k}{m}}$. The other two cases are similar to this, yielding $\lambda = (-1)^{\frac{k}{m}}$, and we omit the calculation.

If $G = A_4, S_4, A_5$, then $k = 12, 24, 60$, respectively. For A_4 , the possible m are $m = 2, 3$, so $k/m = 6, 4$ and $(-1)^{\frac{k}{m}} = 1$. The justification for S_4, A_5 is entirely analogous and we, again, omit the calculation. \square

In the following definitions let $G \subset \mathrm{PGL}_2$ be a finite subgroup.

DEFINITION 7. We say that an orbit of $P \in \mathbb{P}^1$ is *degenerate* if the length of the orbit $|GP|$ is less than the order of the group $|G|$.

DEFINITION 8. We call a pair of effective divisors on \mathbb{P}^1 (D_1, D_2) , where $D_1 = \sum_p s_p p$ and $D_2 = \sum_p t_p p$ a *relevant pair* if:

- (1) $\deg D_2 - \deg D_1 \equiv 2 \pmod{|G|}$
- (2) If H, J are homogeneous forms corresponding to D_1, D_2 respectively, then for the action of every element of the pre-image of G in SL_2 , H and J are eigenvectors with equal eigenvalues.
- (3) For all $p \in \mathbb{P}^1$, $t_p > 1 \implies s_p = 0$.
- (4) For all $p \in \mathbb{P}^1$, $s_p, t_p < |G_p|$.

DEFINITION 9. Similarly, we call an effective divisor $D = \sum t_p p$ a *relevant divisor* if:

- (1) D is fixed by G .
- (2) All t_p are either 0 or 1.
- (3) Every point in $\mathrm{supp}(D)$ has a degenerate orbit.

If D is any effective divisor and G a finite subgroup of PGL_2 let D_G be the effective divisor obtained from H subtracting as many divisors

of the form

$$\sum_{\sigma \in G} P^\sigma$$

as possible.

Let $\bar{g}(\text{Rat}_d)^G = \bar{g}(\text{Rat}_d^G)$ be the fixed set for the action of G on $\bar{g}(\text{Rat}_d)$.

LEMMA 15. *Let G be a finite subgroup of PGL_2 . Let $[(H, J)]$ be in $\bar{g}(\text{Rat}_d)^G$. If $H \neq 0$, then*

$$(\text{div}(H)_G, \text{div}(J)_G)$$

is a relevant pair. Furthermore, if $H = 0$ then $\text{div}(J)_G$ is a relevant divisor.

PROOF. For $H \neq 0$, (1) follows from the fact that H is degree $d-1$, J is degree $d+1$, and any divisors of the form $\sum_{g \in G} gp$ that we subtract in passing to $(\text{div}(H)_G, \text{div}(J)_G)$ have degree $|G|$. For (2), being in $\bar{g}(\text{Rat}_d)^G$ implies that H and J are eigenvectors with equal eigenvalues for the preimage of G in SL_2 , and then by Lemma 14 this is unchanged when removing the linear factors corresponding to the divisors that were subtracted by passing to $(\text{div}(H)_G, \text{div}(J)_G)$. The requirement in (3) follows immediately for $(\text{div}(H), \text{div}(J))$ by Lemma 11, and remains true when passing to $(\text{div}(H)_G, \text{div}(J)_G)$ because any t_p can only get smaller in this process. For (4), first note that the divisors $\text{div}(H)$ and $\text{div}(J)$ are fixed by the action of G . Divisors of the form $\sum_{g \in G} gp$ are fixed by the action of G , and so when we remove them the resulting divisor pair $(\text{div}(H)_G, \text{div}(J)_G)$ is also fixed by G . Note also that if p is in the support of a divisor fixed by G , then clearly the entire orbit of p is in that support as well, since otherwise the divisor would change under the action of G . It is also clear that the coefficients in a divisor must be equal for all points in the same orbit. Thus if any of the coefficients s_p or t_p of $(\text{div}(H)_G, \text{div}(J)_G)$ are greater than or equal to $|G_p|$, it would be possible to subtract a divisor of the form $\sum_{g \in G} gp$, contrary to the definition of D_G .

For $H = 0$, (1) is clear and (2) follows as above from Lemma 11. For (3), if a given p does not have a degenerate orbit, then the divisor $\sum_{g \in G} gp$ has coefficients all equal to 1. Since, as above, the full orbit of p occurs in the divisor $\text{div}(J)$ with equal coefficients, it follows that the entire orbit will be subtracted off in passing to $\text{div}(J)_G$, leaving only degenerate orbits. \square

We have therefore have defined a map

$$S : \bar{g}(\text{Rat}_d)^G \rightarrow \text{relevant pairs} \cup \text{relevant divisors}.$$

PROPOSITION 16. *A relevant pair (D_1, D_2) is in the image of S if and only if*

- (1) $\deg D_1 \leq d - 1$
- (2) $\deg D_2 \leq d + 1$, and $\deg D_2 \equiv d + 1 \pmod{|G|}$

PROOF. Let $(D_1, D_2) = (\text{div}(H)_G, \text{div}(J)_G)$ be in the image of S . Then $\deg(D_1) = \deg(\text{div}(H)_G) \leq \deg(\text{div}(H)) = d - 1$ and similarly, $\deg(D_2) = \deg(\text{div}(J)_G) \leq \deg(\text{div}(J)) = d + 1$. Since D_2 is obtained by subtracting orbits of points from $\text{div}(J)$, we have that $d + 1 = \deg(\text{div}(J)) = \deg(\text{div}(J)_G) + m|G|$, so $\deg(D_2) \equiv d + 1 \pmod{|G|}$.

Conversely, let (D_1, D_2) be a relevant pair of divisors. We aim to construct a pair of forms (H, J) of degree $d - 1$ and $d + 1$, respectively, such that

$$(D_1, D_2) = (\text{div}(H)_G, \text{div}(J)_G).$$

Let H_0 and J_0 be forms associated to D_1 and D_2 . Since (D_1, D_2) is a relevant pair, we have that $\deg(J_0) = \deg(H_0) + 2 + m|G|$ for some m . We also have that, for every action of the pre-image of G in SL_2 , H_0 and J_0 are eigenvectors with equal eigenvalues, and that every multiple zero of J_0 is not a zero of H_0 , and finally that every zero of H_0 or J_0 is degenerate. By condition (2) of this proposition, we have that $\deg(J_0) + \ell|G| = d + 1$. Pick ℓ non-degenerate orbits of \mathbb{P}^1 and let J_1 be the product of these orbits as homogeneous forms. Set $J = J_0 J_1$. It is clear that $\text{div}(J)_G = J_0$. Similarly, $\deg(H_0) + n|G| = d - 1$. Choose n non-degenerate orbits of \mathbb{P}^1 distinct from the previous ℓ ones. Let H_1 be the product of these orbits as homogeneous forms and set $H = H_0 H_1$. Again, $\text{div}(H)_G = H_0$.

That H and J are G -invariant follows by condition (2) of a relevant pair and the construction of H_1 and J_1 . By construction, any multiple zeros potentially added to J will not also be zeros of H , so by Lemma 11, $[(tH, t^{-1}J)]$ is in $\bar{g}(\text{Rat}_d)$ for all but finitely many t . \square

We have a similar description of relevant divisors.

PROPOSITION 17. *A relevant divisor D is in the image of S if and only if*

- (1) $\deg D \leq d + 1$
- (2) $\deg D \equiv d + 1 \pmod{|G|}$

PROOF. This proof is entirely analogous to the proof of Proposition 16 and therefore we omit it. \square

Propositions 16 and 17 now allow us to calculate the dimension of fibers over a relevant pair (D_1, D_2) or a relevant divisor D .

$$\begin{aligned} \dim S^{-1}(D_1, D_2) &= \frac{d-1-\deg D_1}{|G|} + \frac{d+1-\deg D_2}{|G|} + 1 \\ &= \frac{2d - (\deg D_1 + \deg D_2)}{|G|} + 1 \\ &= \left\lfloor \frac{2d}{|G|} \right\rfloor + 1 - \left\lfloor \frac{\deg D_1 + \deg D_2}{|G|} \right\rfloor \end{aligned}$$

Similarly, for a relevant divisor D

$$\dim S^{-1}(D) = \frac{d+1-\deg D}{|G|}$$

PROPOSITION 18. *Fix a finite non-trivial subgroup $G \subset \mathrm{PGL}_2$. Then there are only finitely many relevant pairs and relevant divisors on \mathbb{P}^1 for that group.*

PROOF. Condition (4) in the definition of a relevant pair implies that a relevant pair only involves degenerate orbits, since if $G_p = 1$, it follows that $s_p = t_p = 0$. Degenerate orbits occur only for points that have non-trivial stabilizer, and there are clearly only finitely many such points in \mathbb{P}^1 . We have a bound on the size of the coefficient of each point in our pair of divisors, and so the result follows. The proof for relevant divisors is similar. \square

Since there are finitely many relevant pairs and relevant divisors, it is now clear how to calculate $\dim \mathrm{Rat}_d^G$.

LEMMA 19. *Let G be one of A_4, S_4, A_5 . Then there are 3 degenerate orbits, $2^3 = 8$ relevant divisors and 8 relevant pairs which are of the form $(D_1(D_2), D_2)$ where $D_2 = \sum_p t_p p$ is a relevant divisor and $D_1(D_2) = \sum_p s_p p$ is defined by*

$$s_p = \begin{cases} 0 & \text{if } t_p = 1 \\ |G_p| - 1 & \text{if } t_p = 0. \end{cases}$$

PROOF. The group of characters C of G is cyclic of order 3, 2, 1 for $G = A_4, S_4, A_5$ resp. Let G be one of these groups. Let χ denote a generator for the group of characters. Using geometry of a regular solid and Lemma 13, one sees that degenerate orbits of G and their

corresponding characters are as follows:

	A_4		S_4		A_5	
	<i>size</i>	<i>character</i>	<i>size</i>	<i>character</i>	<i>size</i>	<i>character</i>
(1)	4	χ	8	1	12	1
	6	1	12	χ	30	1
	4	χ^{-1}	6	χ	20	1

Let H be the quotient of Div^G , the group of divisors on \mathbb{P}^1 fixed by G , by the subgroup generated by divisors of the form $\sum_{\sigma \in G} P^\sigma$. H is a finite group. Each divisor fixed by G gives rise to a character and this induces a surjective homomorphism

$$\alpha : H \rightarrow C$$

Let $n = |G|$. The degree of the divisor gives rise to a homomorphism

$$\overline{\deg} : H \rightarrow 2\mathbb{Z}/n\mathbb{Z}$$

Let u be the image in H of the divisor which is the sum of all points in \mathbb{P}^1 with non-trivial stabilizers in G , with all multiplicities equal 1:

$$\sum_{|G_P| \neq \langle 1 \rangle} P \mapsto u \in H$$

From the table above we see that $\deg u = n + 2$ and that $\alpha(u) = 1$. The element u shows that the resulting homomorphism

$$H \rightarrow C \times 2\mathbb{Z}/n\mathbb{Z}$$

is surjective, while counting the orders of the groups shows that this homomorphism is in fact an isomorphism. Pairs of divisors satisfying conditions 1), 2) and 4) of the definition of relevant pairs (Definition 8) correspond bijectively to pairs of the form $(x, x + u)$ where $x \in H$. Imposing the remaining condition 3) on such pairs yields the desired result.

□

THEOREM 20. *Let $d > 1$.*

- (1) $\mathcal{A}(A_4) \neq \emptyset$ if and only if d is odd.
- (2) $\mathcal{A}(S_4) \neq \emptyset$ if and only if d is coprime to 6.
- (3) $\mathcal{A}(A_5) \neq \emptyset$ if and only if d is congruent to one of 1, 11, 19, 29 modulo 30.

Let G be any of these groups and suppose d is such that $\mathcal{A}(G) \neq \emptyset$. Then $\mathcal{A}(G)$ is irreducible and

$$\dim \mathcal{A}(G) = \left\lfloor \frac{2d}{|G|} \right\rfloor.$$

PROOF. In the notation of Lemma 19, $S^{-1}(D_2) \cup S^{-1}(D_1(D_2), D_2)$ is closed and irreducible in Rat_d . For S_4 and A_5 the 8 relevant divisors occur for distinct residues of d modulo $|G|$, while for A_4 it is not the case, but the resulting multiple components are conjugate. The total number of points in the three degenerate orbits of G equals $|G| + 2$. With the aid of this fact, one deduces from the formula above that

$$\left[\frac{\deg D_1(D_2) + \deg D_2}{|G|} \right] = 1$$

for all relevant divisors. That proves the proposition for $d > |G|$. For $d \leq |G|$, the formula is easy to check. \square

COROLLARY 21. *Let G be the group of rotations of a platonic solid, and $d > 1$ such that $A(G) \neq \emptyset$. Then $A(G)$ contains a function with distinct fixed points and all multipliers equal to $d + 1$.*

EXAMPLE 1. Let $d = 5$. Then $\mathcal{A}(A_4) = \mathcal{A}(S_4)$ is a single point in \mathcal{M}_5 given by

$$f(z) = \frac{z^5 - 5z}{1 - 5z^4}.$$

4. Automorphism locus equals singular locus

In this section we combine Luna's slice theorem (cf Drezet [2]) and the dimension calculations of Section 2 to show that the singular locus of \mathcal{M}_d coincides with its automorphism locus for $d > 2$. The impetus for the work presented in this section is an example of a family of singularities in \mathcal{M}_3 in John Milnor's dynamics book ([8]). Effectively, in that example he identifies a Luna etale slice and uses it to find singularities. The following is a variation on a definition suggested by John Milnor in personal correspondence([9]).

DEFINITION 10. We call a point $[\phi] \in \mathcal{M}_d(k)$ *simple* if $\text{Aut}(\phi)$ is cyclic of prime order.

LEMMA 22. *Let p be a prime number and let H be a cyclic group of order p , acting on a vector space V . Suppose $\text{codim}(V^H, V) > 1$. Let $x \in V$. The image of x in the quotient V/H is a singular point precisely when the stabilizer H_x is non-trivial.*

PROOF. The first part of this lemma is the Chevalley-Shephard-Todd theorem (see [1]). We call $g \in H$ a *pseudoreflection* if it fixes a codimension 1 subspace of V and is not the identity transformation. The Chevalley-Shephard-Todd theorem states that for arbitrary finite H (i.e. not necessarily cyclic, prime order), \bar{x} is non-singular if and only if H_x is generated by pseudoreflections. When H is finite cyclic

with $\text{codim}(V^H, V) > 1$ there can be no pseudoreflections, so x will be non-singular if and only if $H_x = 1$. \square

REMARK 4. In fact, over \mathbb{C} , for such x , letting $\ell = 2 \dim V^H + 2$, we have

$$H_\ell(V/H, V/H - \bar{x}) = \mathbb{Z}/p\mathbb{Z}.$$

This implies that the singularities of \mathcal{M}_d are topological, not merely algebraic.

To see this, first translate x to 0 and fix a basis of V and choose the representation of H where H is the cyclic group generated by $g \in \text{GL}_n$ with g is diagonalized and diagonal entries given by p^{th} roots of unity.

Write V as a sum of the fixed subspace and its complement: $V = V^H + V^\perp$. Then V is a sum of H -invariant spaces V^H and V^\perp and the action of H preserves the action of the norm $\|\cdot\|$ on V because the non-zero entries of each matrix in H are roots of unity. Hence, the unit ball maps to itself under the action of H . Let B be the unit ball and S the unit sphere in V . The unit ball is the cone over the sphere, and the action on the sphere is free, so the quotient map $S \rightarrow S/H$ is a covering map.

Let $C(X)$ denote the cone over a variety X with H -action. Then $C(X)/H = C(X/H)$. Since the ball is the cone of the sphere $B/H = C(S)/H = C(S/H)$. Now consider the long exact sequence of relative homology for the cone and the punctured cone. The punctured cone retracts to X and the cone retracts to the vertex. Since our cone is the ball, the punctured cone retracts S and the cone retracts to a point. Since the homology of a point is 0, we have that $H_i(C(X), C(X) - x) = H_{i-1}(X)$ and so $H_i(B, B - x) = H_{i-1}(S) = 0$ or \mathbb{Z} if $i \neq n + 1$ and $i = n + 1$, respectively.

We now that H_1 is the abelianization of the fundamental group, and the fundamental group of S/H is $\mathbb{Z}/p\mathbb{Z}$, so $H_1(S/H) = \mathbb{Z}/p\mathbb{Z}$ and therefore $H_2(B/H, B/H - \bar{x}) = \mathbb{Z}/p\mathbb{Z}$. We now use excision. Assume that $V^H = 0$ Then $H_i(V/H, V/H - \bar{x})$ can be calculated by excising the outside of B/H and we have that $H_i(V/H, V/H - \bar{x}) = H_i(B/H, B/H - \bar{x})$, in particular $H_2(V/H, V/H - \bar{x}) = \mathbb{Z}/p\mathbb{Z}$. If $V^H \neq 0$, then $H_{2 \dim(V^H) + i}(V/H, V/H - \bar{x}) = H_i(B/H, B/H - \bar{x})$ and we see that $H_\ell(V/H, V/H - \bar{x}) = \mathbb{Z}/p\mathbb{Z}$ when $\ell = 2 \dim(V^H) + 2$. We omit the last step of this justification as it takes us too far astray from our original goal.

LEMMA 23. *Let $d > 2$. Simple points of $\mathcal{M}_d(k)$ are singular.*

PROOF. Let $\phi \in \text{Rat}_d$, H be the stabilizer of ϕ , and assume $H = \langle \sigma \rangle$ is cyclic of prime order p . Let N be the normal space at ϕ to the

orbit of ϕ

$$N = T_\phi(\text{Rat}_d)/T_\phi(\text{SL}_2 \cdot \phi).$$

There is an induced action of H on N . Consequently by Luna's slice theorem for smooth varieties, the completion of the local ring at $[\phi]$ in \mathcal{M}_d is isomorphic to the completion of the local ring at the image of 0 in the quotient N/H . Thus we need only to prove that the image of 0 in N/H is a singular point. It follows again from Luna's slice theorem for smooth varieties, that

$$\text{codim}(N^H, N) = \text{codim}(\mathcal{A}_p, \mathcal{M}_d)$$

The latter codimension is at least 2 for $d > 2$, as shown in Section 2. Therefore, by Lemma 22, $[\phi]$ is singular in \mathcal{M}_d . \square

REMARK 5 (Simple points are topologically singular). If $\pi : \text{Rat}_d \rightarrow \mathcal{M}_d$ is the quotient map, then $\pi^{an} : \text{Rat}_d(\mathbb{C}) \rightarrow \mathcal{M}_d(\mathbb{C})$ is a topological quotient with respect to the classical topology (cf Neeman, [12]). Moreover, isomorphism of completions of local rings in the proof above implies that in classical topology, there is a neighborhood of $[\phi]$ in \mathcal{M}_d which is homeomorphic to a neighborhood of the image of 0 in N/H . The image of 0 in N/H is topologically singular according to lemma 22.

LEMMA 24. *Simple points are dense in the automorphism locus \mathcal{A} .*

PROOF. It suffices to show that simple points are dense in \mathcal{A}_p for each prime divisor p of $(d-1)d(d+1)$. Start with \mathcal{A}_2 . Recall that D_m denotes the dihedral group of order $2m$ and consider

$$\mathcal{A}_2 \setminus \left(\bigcup_{m>2} \mathcal{A}_m \cup \mathcal{A}(D_2) \right).$$

It is a dense subset of \mathcal{A}_2 because $\dim \mathcal{A}_m < \dim \mathcal{A}_2$ for $m > 2$ (see Remark 2) and $\dim \mathcal{A}(D_2) < \dim \mathcal{A}_2$ by Proposition 6. It contains only points with stabilizer isomorphic to $\mathbb{Z}/2\mathbb{Z}$, since every platonic solid has a symmetry of order 3.

Let now p be a prime greater than 2. If $\mathcal{A}_p \subseteq \mathcal{A}_2$, simple points with stabilizer isomorphic to $\mathbb{Z}/2\mathbb{Z}$ are already dense in \mathcal{A}_p . Otherwise

$$\mathcal{A}_p \setminus (\mathcal{A}_2 \cup \mathcal{A}_{2p} \cup \mathcal{A}_{3p} \cup \dots)$$

is non-empty, open, and consists of points with stabilizer isomorphic to $\mathbb{Z}/p\mathbb{Z}$. It is non-empty because $\dim \mathcal{A}_{kp} < \dim \mathcal{A}_p$ for $k > 1$ as seen from the formula in Proposition 3. It consists of simple points because dihedral groups and symmetry groups of platonic solids contain an element of order 2. \square

EXAMPLE 2. In the case of degree 2 maps, $\mathcal{A}_3 = \mathcal{A}(D_3)$, hence simple points are not dense in \mathcal{A}_3 . This agrees with the well known result that \mathcal{M}_2 is smooth, but has non trivial automorphism locus.

THEOREM 25. *When $d > 2$, the singular locus of \mathcal{M}_d coincides with the automorphism locus: $\mathcal{A} = \mathcal{S}$.*

PROOF. The singular locus is closed and Lemma 24 shows that $\mathcal{S} \cap \mathcal{A}$ is dense in the automorphism locus, hence \mathcal{A} is contained in the singular locus \mathcal{S} . The converse follows directly from Luna's slice theorem. We conclude that $\mathcal{A} = \mathcal{S}$. \square

5. The Picard and class groups of $\mathcal{M}_d, \mathcal{M}_d^s$, and \mathcal{M}_d^{ss}

In this section we use the results of Section 2 and 4 to compute the Picard and class groups of $\mathcal{M}_d, \mathcal{M}_d^s$, and \mathcal{M}_d^{ss} , where \mathcal{M}_d^s and \mathcal{M}_d^{ss} denote the moduli space of stable and semistable rational maps in the sense of Geometric Invariant Theory (GIT). In this section we work over an algebraically closed field k of characteristic 0. We will freely use terminology from GIT, for which we refer the reader to [10].

LEMMA 26. *Fix $d > 1$ an integer. Then $\text{Pic}(\text{Rat}_d) \cong \mathbb{Z}/2d\mathbb{Z}$.*

PROOF. As $\text{Rat}_d = \mathbb{P}^n - V(\text{Res})$ is an affine open subscheme of projective space it immediately follows that $\text{Pic}(\text{Rat}_d) \cong \text{Cl}(\text{Rat}_d)$ and it suffices to show that $\text{Cl}(\text{Rat}_d) \cong \mathbb{Z}/2d\mathbb{Z}$.

The resultant Res is a homogeneous polynomial of degree $2d$ on \mathbb{P}^{2d+1} . It follows from Proposition 6.5 of [4] that the following sequence is exact.

$$\mathbb{Z} \rightarrow \text{Cl}(\mathbb{P}^{2d+1}) \rightarrow \text{Cl}(\text{Rat}_d) \rightarrow 0$$

where the first map is defined by $1 \mapsto 1 \cdot V(\text{Res})$. Using the well known fact that $\text{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$ for any n , we conclude that $\text{Cl}(\text{Rat}_d) \cong \mathbb{Z}/\deg(\text{Res})\mathbb{Z} = \mathbb{Z}/2d\mathbb{Z}$. \square

In fact, this lemma proves something stronger, that $\text{Pic}(\text{Rat}_d)$ is cyclic order $2d$ generated by $\mathcal{O}(1)|_{\text{Rat}_d}$.

We must next define some terms from GIT.

DEFINITION 11. For G a group variety acting on a variety X via the action $\sigma : G \times X \rightarrow X$ and for any invertible sheaf $\mathcal{L} \in \text{Pic}(X)$ a G -linearization of \mathcal{L} is an isomorphism $\sigma^*\mathcal{L} \rightarrow p_2^*\mathcal{L}$, where p_2 is the projection $G \times X \rightarrow X$, subject to a certain cocycle condition which is equivalent to requiring that the corresponding line bundle is equipped with a G -action which is compatible with the G -action on X .

The collection of G -linearized line bundles on X form a group which we denote by $\text{Pic}^G(X)$. If we fix a G -linearized line sheaf on the variety X we can then describe the largest subset of X which has a geometric quotient.

DEFINITION 12. Let G be an algebraic group acting on a variety X , all over k , and let \mathcal{L} be a G -linearized line sheaf. Let $x \in X$ be a geometric point. We call x *semi-stable* if there exists a section $s \in H^0(X, \mathcal{L}^n)$ for some n , such that $s(x) \neq 0$, the fiber X_s is affine, s is invariant. If, furthermore, the action of G on X_s is closed, then we call x *stable*.

We let X^s denote the stable locus for the action of G on X and fixed linearization, and X^{ss} the semi-stable locus. It is clear from the definition that these are open subsets. In this paper, $(\mathbb{P}^{2d+1})^s$ and $(\mathbb{P}^{2d+1})^{ss}$ refer to the stable and semi-stable locus for the SL_2 -linearization of $\mathcal{O}(1)$ on \mathbb{P}^{2d+1} for the SL_2 -action of conjugation.

LEMMA 27. Fix $d > 1$ an integer and let $(\mathbb{P}^{2d+1})^s$ and $(\mathbb{P}^{2d+1})^{ss}$ denote the stable and semi-stable locus for the SL_2 -action of conjugation on \mathbb{P}^{2d+1} . Then the open immersions $(\mathbb{P}^{2d+1})^s \subset (\mathbb{P}^{2d+1})^{ss} \subset \mathbb{P}^{2d+1}$ induce isomorphisms $\text{Pic}((\mathbb{P}^{2d+1})^s) \cong \text{Pic}((\mathbb{P}^{2d+1})^{ss}) \cong \text{Pic}\mathbb{P}^{2d+1} \cong \mathbb{Z}$.

PROOF. Let $Z^s = \mathbb{P}^{2d+1} - (\mathbb{P}^{2d+1})^s$ and $Z^{ss} = \mathbb{P}^{2d+1} - (\mathbb{P}^{2d+1})^{ss}$ be the unstable, and not semi-stable loci, respectively. As the stable and semi-stable loci are open, it follows that Z^s and Z^{ss} are closed subsets of \mathbb{P}^{2d+1} . Furthermore, $Z^s, Z^{ss} \subset V(\text{Res})$ are proper closed subsets of the irreducible closed subset $V(\text{Res})$. Consequently $\text{codim}(Z^s, \mathbb{P}^{2d+1}) > 1$ and $\text{codim}(Z^{ss}, \mathbb{P}^{2d+1}) > 1$. It follows from Proposition 6.5 of [4] that $\text{Pic}((\mathbb{P}^{2d+1})^s) \cong \text{Pic}(\mathbb{P}^{2d+1}) \cong \mathbb{Z}$ and similarly $\text{Pic}((\mathbb{P}^{2d+1})^{ss}) \cong \mathbb{Z}$. \square

Let $A \subset \mathbb{P}^{2d+1}$ be the subset of points of \mathbb{P}^{2d+1} which have a non-trivial stabilizer for the PGL_2 -action of conjugation. A is closed in $(\mathbb{P}^{2d+1})^{ss}$ as a consequence of Luna's slice theorem.

DEFINITION 13. The set $\mathbb{P}_*^{2d+1} = \mathbb{P}^{2d+1} - A$ denotes the *free locus* of \mathbb{P}^{2d+1} for the PGL_2 -action of conjugation. For any subvariety of $X \subset \mathbb{P}^{2d+1}$ for which the PGL_2 action restricts, we use the lower star to indicate the intersection with the free locus: $X_* = X \cap \mathbb{P}_*^{2d+1}$.

It immediately follows from the definition that for any subvariety of the semistable locus, $X \subseteq (\mathbb{P}^{2d+1})^{ss}$, X_* is an open subvariety of X and the restricted action of PGL_2 on X_* is free.

PROPOSITION 28. *Let $d > 1$ be an integer. Then*

$$\begin{aligned}\mathrm{Pic}(\mathrm{Rat}_{d*}) &\cong \mathrm{Pic}(\mathrm{Rat}_d), \\ \mathrm{Pic}((\mathbb{P}^{2d+1})_*^s) &\cong \mathrm{Pic}((\mathbb{P}^{2d+1})^s), \\ \mathrm{Pic}((\mathbb{P}^{2d+1})_*^{ss}) &\cong \mathrm{Pic}((\mathbb{P}^{2d+1})^{ss}).\end{aligned}$$

PROOF. Since the varieties involved are non-singular, class group is isomorphic to Picard group, and the proof follows immediately using Proposition 7 on the codimension of the automorphism locus in \mathbb{P}^{2d+1} and Proposition 6.5 of [4]. \square

We will use the following Descent lemma from [11] (Theoreme 2.3).

THEOREM 29 (Descent Lemma). *Let $\pi : X \rightarrow Y$ be a good quotient. Let F be a G -vector bundle on X . Then F descends to Y if and only if for all closed points x of X such that the orbit Gx is closed, the stabilizer of x in G acts trivially on the fiber F_x .*

COROLLARY 30. *There is a natural injection*

$$\pi^* : \mathrm{Pic}(Y) \rightarrow \mathrm{Pic}^G(X),$$

whose image consists of all isomorphism classes of G -linearized line bundles L on X with the following property:

For every closed point x in X for which the orbit Gx is closed, the stabilizer of x in G acts trivially on the fiber L_x .

PROOF. It is easy to show that the map $\pi^* : \mathrm{Pic}(Y) \rightarrow \mathrm{Pic}(X)$ is injective when π is a good quotient. Furthermore, $\mathrm{Pic}(X) \rightarrow \mathrm{Pic}^G(X)$ is injective, proving the Corollary. \square

PROPOSITION 31. *Let X be a normal irreducible variety over k with SL_2 -action. Then any $\mathcal{L} \in \mathrm{Pic}(X)$ admits a unique SL_2 -linearization.*

PROOF. By Theorem 7.2 of [3], the natural map $\alpha : \mathrm{Pic}^{SL_2}(X) \rightarrow \mathrm{Pic}(X)$ fits into an exact sequence

$$0 \rightarrow \mathrm{Ker}(\alpha) \rightarrow \mathrm{Pic}^{SL_2}(X) \xrightarrow{\alpha} \mathrm{Pic}(X) \rightarrow \mathrm{Pic}(SL_2)$$

$\mathrm{Pic}(SL_2)$ is known to be trivial and consequently α is surjective. $\mathrm{Ker}(\alpha)$ consists of the isomorphism classes of SL_2 -linearizations on the trivial bundle $X \times \mathbb{A}^1$. In fact, by Corollary 7.1 of [3], $\mathrm{Ker}(\alpha) \cong \mathrm{Hom}_d^n(SL_2, \mathbb{G}_m)$, i.e. that $\mathrm{Ker}(\alpha)$ is isomorphic to the character group of SL_2 . The character group of a connected affine algebraic group which is complete is trivial, so $\mathrm{Ker}(\alpha) = 1$ and the proposition is finished. \square

THEOREM 32. *In the notation above, there is a natural isomorphism between Rat_{d*}/SL_2 and $\mathcal{M}_{d,*} = \mathcal{M}_d - \mathcal{A}$.*

PROOF. This follows from basic facts of GIT. \mathcal{M}_d is a geometric quotient. Rat_{d*} is an open, SL_2 invariant subset of Rat_d , so it also has a geometric quotient.

In general if G acts on X and $\mathcal{L} \in \text{Pic}^G(X)$, then if $U, V \subset X$ are open sets such that $V \subset U \subset X^s(\mathcal{L})$, then $V/G \subset U/G$ is an open subset of the quotient, both being geometric.

It follows that $\text{Rat}_{d*}/\text{SL}_2 \subset \mathcal{M}_d$ is an open subvariety where all points have trivial stabilizer. Moreover, it is the largest such open subvariety. It therefore coincides with $\mathcal{M}_{d,*} = \mathcal{M}_d - \mathcal{A}$. \square

PROPOSITION 33.

- (1) $\text{Cl}(\mathcal{M}_d)$ is as follows: $\mathbb{Z}/2d\mathbb{Z}$ when d is odd and $\mathbb{Z}/d\mathbb{Z}$ when d is even.
- (2) $\text{Cl}(\mathcal{M}_d^s) = \text{Cl}(\mathcal{M}_d^{ss}) \cong \mathbb{Z}$

PROOF. (1) Note first that, because Rat_d is a normal scheme, \mathcal{M}_d is also normal (see proposition 3.1 p. 45 of [3]) and so Proposition 6.5 of [4] applies. We can thus conclude that $\text{Cl}(\mathcal{M}_d) = \text{Cl}(\mathcal{M}_{d*})$. It follows from the previous theorems that \mathcal{M}_{d*} is a good (in fact geometric) quotient of Rat_{d*} by SL_2 and by Descent lemma (Theorem 29) the image of $\text{Pic}(\mathcal{M}_d)$ in $\text{Pic}^{\text{SL}_2}(\text{Rat}_{d*})$ consists of isomorphism classes of linearized line bundles L for which $\langle \pm I \rangle$ acts trivially on the fiber of L at every closed point of Rat_{d*} . By lemma 8 the action of $-I$ on $\mathcal{O}_{\mathbb{P}^{2d+1}}(1)$ is trivial for d odd and is via multiplication by -1 throughout when d is even.

Rat_{d*} is an open subscheme of projective space and is therefore normal. By Proposition 31 it follows that $\text{Pic}^{\text{SL}_2}(\text{Rat}_{d*}) = \text{Pic}(\text{Rat}_{d*}) = \mathbb{Z}/2d\mathbb{Z}$. And our calculation shows that the image of $\text{Pic}(\mathcal{M}_d)$ in $\text{Pic}(\text{Rat}_{d*})$ is generated by $\mathcal{O}(1)$, when d is odd and by $\mathcal{O}(2)$ when d is even. We conclude that $\text{Pic}(\mathcal{M}_{d*}) = \mathbb{Z}/2d\mathbb{Z}$ for odd d and $\text{Pic}(\mathcal{M}_{d*}) = \mathbb{Z}/d\mathbb{Z}$ for even d . Finally, since \mathcal{M}_{d*} is non-singular (Proposition 25), $\text{Pic}(\mathcal{M}_{d*}) = \text{Cl}(\mathcal{M}_{d*})$, and since codimension of its complement in \mathcal{M}_d is greater than 1, $\text{Cl}(\mathcal{M}_{d*}) = \text{Cl}(\mathcal{M}_d)$.

(2) The proof is nearly identical for \mathcal{M}_d^s . This approach requires more care for \mathcal{M}_d^{ss} as \mathcal{M}_d^{ss} is not a geometric quotient. Regardless, \mathcal{M}_d^s is a dense open subset of \mathcal{M}_d^{ss} and therefore $\mathcal{M}_d^s = \mathcal{M}_d^{ss} - V$ for some close subset of codimension $\text{codim}(V, \mathcal{M}_d^{ss}) > 1$. It follows that $\text{Cl}(\mathcal{M}_d^{ss}) = \text{Cl}(\mathcal{M}_d^s) = \mathbb{Z}$. \square

THEOREM 34. Let $d > 1$ be an integer. Then $\text{Pic}(\mathcal{M}_d)$ is trivial.

PROOF. By the descent lemma (Theorem 29), $\text{Pic}(\mathcal{M}_d)$ can be identified with those elements L of

$$\text{Pic}^{\text{SL}_2}(\text{Rat}_d) = \text{Pic}(\text{Rat}_d) = \mathbb{Z}\mathcal{O}(1)/\mathbb{Z}\mathcal{O}(2d)$$

for which the stabilizer of every function acts trivially on the corresponding fiber of L . We know that Rat_d contains functions ϕ with automorphisms $\sigma \in \text{PGL}_2$ of order m as long as $m|d$ or $m|d \pm 1$. Let g represent σ in SL_2 . Then g acts on the stalk of $\mathcal{O}(1)$ at ϕ by multiplication by s -th root of unity, where s is described in Corollary 9.

Case d even. Take $m = d$. Then $s = 2d$, hence only line bundles $\mathcal{O}(2dk)$ descend to \mathcal{M}_d , but these are all trivial.

Case d odd. Take $m = d$ again. Then $s = d$. Then take $m = d + 1$ or $m = d - 1$. Then $s = 2$. Again, only line bundles $\mathcal{O}(2dk)$ descend to \mathcal{M}_d , and these are all trivial. \square

LEMMA 35. *Let $x \in (\mathbb{P}^{2d+1})^{ss}$ be a closed point. Then $x \in (\mathbb{P}^{2d+1})^s$ if and only if its orbit is closed in $(\mathbb{P}^{2d+1})^{ss}$ and its stabilizer is of finite order. Moreover, the order of the stabilizer of x is bounded as x runs through $(\mathbb{P}^{2d+1})^s$.*

PROOF. For the first part, see (cf [10], Amplification 1.11). For the second, consider the quasi-finite maps

$$\text{SL}_2 \times (\mathbb{P}^{2d+1})^s \rightarrow (\mathbb{P}^{2d+1})^s \times (\mathbb{P}^{2d+1})^s$$

given by

$$(g, x) \mapsto (x, gx).$$

The stabilizers then correspond to the fibers over the diagonal, and quasi-finite maps of varieties have bounded fiber cardinality. \square

THEOREM 36. *Let $d > 1$ be an integer. The open immersion $\mathcal{M}_d^s \subset \mathcal{M}_d^{ss}$ induces an isomorphism $\text{Pic} \mathcal{M}_d^{ss} \cong \text{Pic} \mathcal{M}_d^s \cong \mathbb{Z}$.*

PROOF. Recall that $\text{Pic}((\mathbb{P}^{2d+1})^s) = \text{Pic}((\mathbb{P}^{2d+1})^{ss})$ (cf Lemma 27) is infinite cyclic generated by $\mathcal{O}(1)$.

Let us look at $\text{Pic}(\mathcal{M}_d^s)$ first. Denote by M a common multiple of orders of stabilizers of points in \mathbb{P}^{2d+1s} . Such M exists by lemma 35. Then $\mathcal{O}(M)$ descends to the quotient by Descent lemma 29 and since the Picard subgroup of the quotient can be identified with line bundles which descend to the quotient, the conclusion follows.

Now let us look at $\text{Pic}(\mathcal{M}_d^{ss})$ and assume that $d = 2m + 1$, since for even d , $\mathcal{M}_d^{ss} = \mathcal{M}_d^s$. Let $x \in \text{Pic}((\mathbb{P}^{2d+1})^{ss}) \setminus \text{Pic}((\mathbb{P}^{2d+1})^s)$ have a closed orbit. Then the stabilizer of x is infinite by Lemma 35. Direct computation using the numerical criterion (cf [13], Theorem 4.40) shows that x is conjugate to either $(X^{m+1}Y^m : 0)$ or $(cX^{m+1}Y^m : X^mY^{m+1})$ with $\{f(z) = cz | c \neq 0\}$ as the stabilizer within PGL_2 in both cases. Using Lemma 8 one can see that in both cases the stabilizer within SL_2 acts trivially on the fiber of the line bundle $\mathcal{O}(1)$ at x . Hence such a point x does not obstruct the descent of line bundles and thus a line bundle

over $(\mathbb{P}^{2d+1})^{ss}$ descends to the quotient if and only if its restriction to $(\mathbb{P}^{2d+1})^s$ does. \square

References

- [1] CHAVALLEY, C. *Invariants of finite groups generated by reflections*, Amer. J. of Math. Vol. 77 Iss. 4 (1955) 778–782.
- [2] DREZET, J.-M., *Luna’s Slice Theorem And Applications*, 2007.
- [3] DOLGACHEV, I. *Lectures on Invariant Theory*, 296 of *London Mathematical Society lecture note series*. Cambridge University Press, Cambridge, 2003.
- [4] HARTSHORNE, R. *Algebraic Geometry*, vol. 52 of *Graduate Texts in Mathematics*. Springer, New York, NY 10010.
- [5] KLEIN, F. *Lectures on the Icosahedron and the Solution of Equations of the Fifth Degree*, Translated by George Gavin Morrice. Dover, New York, 1956 revised ed.
- [6] LEVY, A. *The space of morphisms on projective space*. *Acta Arith.* Vol. 146 Iss. 1 (2011), 13–31.
- [7] LONSTED, K. *The singular points on the moduli space for smooth curves*, *Math. Ann.* Vol. 266, (1984), 397–402.
- [8] MILNOR, J. *Dynamics of One Complex Variable (AM-160): Third Edition*. Princeton University Press, Princeton, NJ. 2011.
- [9] MILNOR, J. *Personal communication with the authors, 2013*.
- [10] MUMFORD, D. *Geometric Invariant Theory*, vol. -34 of *Ergebnisse Der Mathematik Un Ihrer Grenzgebiete*. 1st edition. Springer-Verlang, Berlin.
- [11] NARASIMHAN, M.S., DREZET, J.-M. *Groupe de Picard des varietes de modules de fibres semi-stables sur les courbes algebriques*. *Inv. math.* vol. 97 (1989), 53–94.
- [12] NEEMAN, A. *The Topology of Quotient Varieties*. *Ann. of Math.* Vol. 122, No. 2 (Sep., 1985), pp 419–459
- [13] SILVERMAN, J. H. *The Arithmetic of Dynamical Systems*, vol. 241 of *Graduate Texts in Mathematics*. Springer, New York, NY 10010. 2007.
- [14] SILVERMAN, J. H. *Moduli Spaces and Arithmetic Dynamics*, vol. 30 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2012.
- [15] SILVERMAN, J. H. *The space of rational maps on \mathbb{P}^1* . *Duke Mat J.* Vol. 94 Iss. 1 (1998), 41–77.
- [16] WEST, L. *Personal communication with the authors, 2013*.

NIKITA MIASNIKOV; PH.D. PROGRAM IN MATHEMATICS; CUNY GRADUATE CENTER; 365 FIFTH AVENUE; NEW YORK, NY 10016 U.S.A.

E-mail address: nmiasnikov@gc.cuny.edu

BRIAN STOUT; DEPARTMENT OF MATHEMATICS; UNITED STATES NAVAL ACADEMY; ANNAPOLIS, MD 21401 U.S.A.

E-mail address: stout@usna.edu

PHILLIP WILLIAMS; DEPARTMENT OF MATHEMATICS; THE KING'S COLLEGE;
NEW YORK, NY 10004 U.S.A.

E-mail address: `pwilliams@tkc.edu`